An Extension in the PD Controller Case of an Asymptotic Stability Theorem of AQM Supporting Nonlinear TCP Flows

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Abstract. Using the fluid-flow nonlinear model of TCP dynamics, an extension in the PD controller case of Theorem 2 presented in (Hollot and Chait, 2001) is proved.

Keywords: Fluid-flow TCP model, PD controller, asymptotic stability

1 Introduction

The main problem of the Internet traffic, needing theoretical/practical solutions, is to prevent its congestion. The mechanism for feedback traffic control on the Internet is the Active Queue Management (AQM) for Transmission Control Protocol (TCP) flows. The variable transmission delays in Internet traffic have a negative impact on the feedback control stability and, consequently, AQM/TCP does not generally work as well as when it was designed. Many researchers have addressed these problems in different ways (Mascolo, 1999; Gunnarsson, 2000; Mascolo, 2000; Hollot *et al*., 2001; Hollot and Chait, 2001; Hollot *et al*., 2002; Low *et al*., 2002; Ryu *et al*., 2003; Cela *et al*., 2005; Rafe' *et al*., 2007; Al-Hammouri, 2008; De Cicco *et al*., 2011; Tolaimate and Elalami, 2011; Voicu, 2012). Their main objective is to maintain local asymptotic stability for arbitrary network delays, link capacities, and routing topologies.

In this note, using the fluid-flow nonlinear model of TCP dynamics developed in (Misra *et al.*, 2000), it is proved an extension in the PD controller case of Theorem 2 given in (Hollot and Chait, 2001).

2 Fluid-Flow Models of TCP Behaviour

The dynamic behaviour of TCP according to the fluid-flow model (Misra et al., 2000; Hollot and Chait, 2001; Hollot et al., 2002) is described by the following system of nonlinear time-variant differential equations:

$$\begin{cases} \dot{w}(t) = -\dfrac{w(t)}{2}\dfrac{w(t-r(t))}{r(t-r(t))}p(t-r(t)) + \dfrac{1}{r(t)} \\ \dot{q}(t) = \dfrac{n(t)}{r(t)}w(t) - c(t), \end{cases} \quad (1)$$

with

$$r(t) = T_p + \frac{q(t)}{c(t)}, \quad (2)$$

where

$w(t) \in [0, \bar{w}]$ (with $\bar{w} = \text{const.} > 0$) is the average TCP window size (in packets),

$q(t) \in [0, \bar{q}]$ (with $\bar{q} = \text{const.} > 0$) is the average queue length (in packets),

$r(t) \in [0, \bar{r}]$ (with $\bar{r} = \text{const.} > 0$) is the round-trip (delay) time,

$c(t) \in [0, \bar{c}]$ (with $\bar{c} = \text{const.} > 0$) is the link capacity,

$n(t) \in [0, \bar{n}]$ (with $\bar{n} = \text{const.} > 0$) is the load factor,

$p(t) \in [0,1]$ is the probability of packets mark, and

$T_p > 0$ is the propagation delay.

3 Asyptotic Stability Theorem for PD Controller

In order to obtain the system of nonlinear time-invariant equations associated to (1), we take into account the following hypotheses:

1^o Around the operating point (w_0, q_0, p_0) we may consider: $n(t) = N = \text{const.}$, $c(t) = C = \text{const.}$

2^o If $r(t)$ appears as an argument of a function, then we consider: $r(t) = R = \text{const.}$

Now, from model (1) we obtain the following system:

$$\begin{cases} \dot{w}(t) = -\dfrac{w(t)}{2}\dfrac{w(t-R)}{R}p(t-R) + \dfrac{1}{R} \\ \dot{q}(t) = \dfrac{N}{R}w(t) - C. \end{cases} \quad (3)$$

Following (Hollot and Chait, 2001; Hollot *et a.l*, 2002), the right hand side of $\dot{w}(t)$ in (3) may be approximated by:

$$-\frac{w(t)}{2}\frac{w(t-R)}{R}p(t-R) + \frac{1}{R} \cong -\frac{w^2(t)}{2R}p(t-R) + \frac{1}{R}. \quad (4)$$

At the same time, with (4), for the operating point we may write:

$$\dot{q} = 0 \Rightarrow w_0 = \frac{RC}{N} \geq 0, \quad (5)$$

$$\dot{w}=0 \Rightarrow w_0^2 p_0 = 2 \Rightarrow p_0 = \frac{2N^2}{C^2R^2} \geq 0. \qquad (6)$$

Now, denoting by

$$\tilde{w} = w - w_0,\ \tilde{q} = q - q_0,\ \tilde{p} = p - p_0 \qquad (7)$$

the small variations of the state variables (w,q) and of the control p, all about the operating point (w_0, q_0, p_0), we obtain the following system of equations:

$$\begin{cases} \dot{\tilde{w}}(t) = -\dfrac{1}{2R}\left\{[\tilde{w}(t)+w_0]^2\,\tilde{p}(t-R) - \left[\tilde{w}^2(t)+2w_0\tilde{w}(t)\right]p_0\right\} \\ \dot{\tilde{q}}(t) = \dfrac{N}{R}\tilde{w}(t). \end{cases} \qquad (8)$$

According to the purpose of this note, we consider now the PD controller which is described by the equation:

$$p(t) = k_1 q(t) + k_2 \dot{q}(t), \quad (9)$$

with $k_1 > 0,\ k_2 \geq 0$ and

$$p_0 = k_1 q_0, \qquad (10)$$

$$\tilde{p}(t) = k_1\tilde{q}(t) + k_2\dot{\tilde{q}}(t). \quad (11)$$

Equations (8) and (11) describe the functioning of the entire system in the case of a PD control structure for AQM/TCP flows. Eliminating $\tilde{w}(t)$ and $\tilde{p}(t)$ between equations (8) and (11) it results

$$\begin{aligned} &\ddot{\tilde{q}}(t) + \beta b(\dot{\tilde{q}}(t))k_1[\tilde{q}(t-R) + q_0] + \\ &\quad +[\beta b(\dot{\tilde{q}}(t)) + \alpha]k_2\dot{\tilde{q}}(t-R) + \alpha k_1\tilde{q}(t-R) = 0, \end{aligned} \qquad (12)$$

where

$$\alpha = \frac{C^2}{2N},\ \ \beta = \frac{1}{N},\ \ b(\dot{\tilde{q}}(t)) = 0,5\dot{\tilde{q}}^2(t) + C\dot{\tilde{q}}(t). \qquad (13)$$

Theorem (extension of Theorem 2, Hollot and Chait, 2001)

For sufficiently small $\beta(k_1+\alpha^{1/2}k_2) > 0$, *the equilibrium point of system* (8), (11) *is asymptotically stable. A region of attraction is given by* (26) *and* (27).

Proof

Consider the following positive definite Lyapunov function:

$$V(\tilde{q},\dot{\tilde{q}}) = 0,5\dot{\tilde{q}}^2(t) + 0,5\alpha k_1\tilde{q}^2(t)\,. \ (14)$$

The time derivative of $V(\tilde{q},\dot{\tilde{q}})$ is

$$\dot{V}(\tilde{q},\dot{\tilde{q}}) = \dot{\tilde{q}}\ddot{\tilde{q}} + \alpha k_1 \tilde{q}\dot{\tilde{q}} = \dot{\tilde{q}}(\ddot{\tilde{q}} + \alpha k_1 \tilde{q}), \qquad (15)$$

where, for the simplicity, the argument t is no more written.

On the other hand, from (12) it follows that:

$$\ddot{\tilde{q}} + \alpha k_1 \tilde{q}(t-R) = -\beta b(\dot{\tilde{q}}) k_1 [\tilde{q}(t-R) + q_0] - \\ -[\beta b(\dot{\tilde{q}}) + \alpha] k_2 \dot{\tilde{q}}(t-R) = 0. \qquad (16)$$

Using the identities:

$$\tilde{q}(t-R) = \tilde{q}(t) - \int_{-R}^{0} \dot{\tilde{q}}(t+\vartheta) d\vartheta, \quad (17)$$

$$\dot{\tilde{q}}(t-R) = \dot{\tilde{q}}(t) - \int_{-R}^{0} \ddot{\tilde{q}}(t+\vartheta) d\vartheta, \quad (18)$$

in (16), we may write successively:

$$\ddot{\tilde{q}} + \alpha k_1 [\tilde{q} - \int_{-R}^{0} \dot{\tilde{q}}(t+\vartheta) d\vartheta] = \\ = -\beta b(\dot{\tilde{q}}) k_1 [\tilde{q} - \int_{-R}^{0} \dot{\tilde{q}}(t+\vartheta) d\vartheta + q_0] - \\ -[\beta b(\dot{\tilde{q}}) + \alpha] k_2 [\dot{\tilde{q}} - \int_{-R}^{0} \ddot{\tilde{q}}(t+\vartheta) d\vartheta],$$

$$\ddot{\tilde{q}} + \alpha k_1 \tilde{q} = -\beta b(\dot{\tilde{q}}) k_1 [\tilde{q} + q_0] - [\beta b(\dot{\tilde{q}}) + \alpha] k_2 \dot{\tilde{q}} + \\ +[\beta b(\dot{\tilde{q}}) + \alpha][k_1 \int_{-R}^{0} \dot{\tilde{q}}(t+\vartheta) d\vartheta + \\ + k_2 \int_{-R}^{0} \ddot{\tilde{q}}(t+\vartheta) d\vartheta]. \qquad (19)$$

By replacing (19) into (15) it results:

$$\dot{V}(\tilde{q},\dot{\tilde{q}}) = -\beta b(\dot{\tilde{q}}) k_1 (\tilde{q} + q_0)\dot{\tilde{q}} - [\beta b(\dot{\tilde{q}}) + \alpha] k_2 \dot{\tilde{q}}^2 + \\ +[\beta b(\dot{\tilde{q}}) + \alpha][k_1 \int_{-R}^{0} \dot{\tilde{q}}(t+\vartheta) d\vartheta + \\ + k_2 \int_{-R}^{0} \ddot{\tilde{q}}(t+\vartheta) d\vartheta]\dot{\tilde{q}}. \qquad (20)$$

Now we may assume a $\lambda > 1$ such that

$$V(\xi) \le \lambda V(t) \quad (21)$$

for all $t - R \le \xi \le t$. Inequality (21) implies:

$$\dot{\tilde{q}}(\xi) \le \lambda \|z(t)\|, \;\; \ddot{\tilde{q}}(\xi) \le \lambda \alpha^{1/2} \|z(t)\| \qquad (22)$$

for all $t - R \le \xi \le t$, where $z(t) = [\alpha^{1/2}\tilde{q}(t) \;\; \dot{\tilde{q}}(t) \;\; \alpha^{-1/2}\ddot{\tilde{q}}(t)]^T$. According to (22) it follows that:

$$k_1 \int_{-R}^{0} \dot{\tilde{q}}(t+\vartheta) d\vartheta + k_2 \int_{-R}^{0} \ddot{\tilde{q}}(t+\vartheta) d\vartheta \le \\ \le \lambda R (k_1 + \alpha^{1/2} k_2) \|z(t)\| \qquad (23)$$

for all $t-R \le \xi \le t$. Substituting (23) into (20) it results:

$$\begin{aligned}\dot V(\tilde q,\dot{\tilde q}) \le & -b(\dot{\tilde q})k_1\beta(\tilde q+q_0)\dot{\tilde q}-[\beta b(\dot{\tilde q})+\alpha]k_2\dot{\tilde q}^2+\\ & +[\beta b(\dot{\tilde q})+\alpha][\lambda R(k_1+\alpha^{1/2}k_2)\|z(t)\|]\dot{\tilde q},\\ \dot V(\tilde q,\dot{\tilde q}) \le & -b(\dot{\tilde q})\beta[k_1(\tilde q+q_0)-\lambda R(k_1+\alpha^{1/2}k_2)\|z\|]\dot{\tilde q}-\\ & -[\beta b(\dot{\tilde q})+\alpha]k_2\dot{\tilde q}^2+\lambda R\alpha(k_1+\alpha^{1/2}k_2)\|z\|\dot{\tilde q}.\end{aligned} \quad (24)$$

Since $\tilde q$ and $\dot{\tilde q}$ are bounded, it is possible to take $\alpha(k_1+\alpha^{1/2}k_2)=\beta(k_1+\alpha^{1/2}k_2)$ small enough to force $(k_1+\alpha^{1/2}k_2)\|z\| \approx (k_1+\alpha^{1/2}k_2)\left|\dot{\tilde q}\right|$ and to make negligible the last term on the right-hand side of (24). It results that:

$$\begin{aligned}\dot V(\tilde q,\dot{\tilde q}) \le & -\beta b(\dot{\tilde q})[k_1(\tilde q+q_0)-\lambda R(k_1+\alpha^{1/2}k_2)\left|\dot{\tilde q}\right|]\dot{\tilde q}-\\ & -[\beta b(\dot{\tilde q})+\alpha]k_2\dot{\tilde q}^2.\end{aligned} \quad (25)$$

According to (Hollot and Chait, 2001) it follows that there exists a $(\tilde q,\, \dot{\tilde q})$ - neighborhood over which $\dot V(\tilde q,\, \dot{\tilde q})<0$ and a region of attraction is level-set interior

$$V_L=\{(\tilde q,\dot{\tilde q}):V_L(\tilde q,\dot{\tilde q})<L\}, \quad (26)$$

where L satisfies

$$(\tilde q,\dot{\tilde q})\in V_L \Rightarrow k_1(\tilde q+q_0)-\lambda R(k_1+\alpha^{1/2}k_2)\left|\dot{\tilde q}\right|>0. \quad (27)$$

4 Conclusion

In this note, using the fluid-flow nonlinear model of TCP dynamics, it is proved an extension in the PD controller case of Theorem 2 given in (Hollot and Chait, 2001).

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